THE ASYMPTOTIC EXPANSION OF THE STURM-LIOUVILLE FUNCTIONS.

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1. In the Proceedings of the National Academy of Sciences (vol. 3, pp. 656–659), Professor Birkhoff gave a direct proof of the closure of the set of Sturm-Liouville functions defined by the equation and boundary conditions,

$$\frac{d^2y}{dx^2} + [\rho^2 - g(x)]y = 0, \qquad y(0) = y(1) = 0. \tag{1}$$

This proof was based on a certain asymptotic expansion for these functions, similar to those used by Hobson* and Kneser.† At the suggestion of Professor Birkhoff I have undertaken to develop this expansion in detail, using explicitly the Volterra integral equation of the second kind used more or less implicitly by most writers in this connection; the method of successive approximation employed in the asymptotic development of the characteristic numbers is capable of extension to the functions satisfying the boundary conditions

$$\alpha' y(0) - \alpha y'(0) = 0, \qquad \alpha\alpha' \geqq 0,$$

$$\beta' y(0) + \beta y'(0) = 0, \qquad \beta\beta' \geqq 0.$$

The explicit use of the Volterra integral equation is especially convenient in the study of the differentiability of the characteristic functions with respect to a parameter σ, introduced by replacing $g(x)$ by $\sigma g(x)$.

1. **Some preliminary inequalities.** Assume that $g(x)$ has bounded variation; instead of the system (1) consider first the system

$$\frac{d^2y}{dx^2} + [\rho^2 - \sigma g(x)]y = 0, \qquad y(0) = y(1) = 0. \tag{2}$$

The equation above can be written in the form

$$\frac{d^2y}{dx^2} + \rho^2 y = \sigma g(x)y,$$

which leads to the Volterra integral equation of the second kind,

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* Proceedings of the London Math. Soc., ser. (2), vol. 6, p. 374.

† Die Integralgleiehungen und ihre Anwendung in der mathematischen Physik, Chap. 3.

$$y(x) = \alpha \sin \rho(x - \beta) + \frac{\sigma}{\rho}\int_0^x \sin \rho(x - \xi)g(\xi)y(\xi)d\xi. \tag{3}$$

Here α, β are arbitrary constants; if $y(x)$ satisfies the boundary condition $y(0) = 0$, we may assume $\beta = 0$. For convenience assume α positive or zero; ρ, σ are real, and $g(x)$ is real for $0 \leq x \leq 1$. Since $g(x)$ has bounded variation for $0 \leq x \leq 1$, $|g(x)|$ has an upper bound G in this interval.

Suppose

$$K_\rho(x_1\xi) = \sin \rho(x - \xi)g(\xi), \qquad |K_\rho(x_1\xi)| \leq G.$$

Equation (3) becomes,

$$y(x) = \alpha \sin \rho x + \frac{\sigma}{\rho}\int_0^x K_\rho(x, \xi)y(\xi)d\xi. \tag{4}$$

Suppose

$$U_\nu = \left(\frac{\sigma}{\rho}\right)^\nu \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{\nu-1}} K_\rho(x, \xi_1)K_\rho(\xi_1, \xi_2) \cdots K_\rho(\xi_{\nu-1}, \xi_\nu) \sin \rho\xi_\nu d\xi_1 d\xi_2 \cdots d\xi_\nu. \tag{5}$$

Then the solution of (4) can be given in the form

$$y(x) = \alpha\left\{\sin \rho x + \sum_{\nu=1}^{\infty} U_\nu(x)\right\}, \tag{6}$$

and this series is dominated by the series

$$\alpha\left\{1 + \sum_{\nu=1}^{\infty}\frac{1}{\nu!}\left(\frac{\sigma G x}{\rho}\right)^\nu\right\} = \alpha e^{\sigma G x/\rho}. \tag{7}$$

For convenience assume $\alpha = 1$. From (5), (6) it is seen immediately that for large values of $|\rho|$ the first few terms of the series are the important ones. Expand U_1:

$$U_1 = \frac{\sigma}{\rho}\int_0^x \sin \rho(x - \xi_1) \sin \rho\xi_1 g(\xi_1)d\xi_1.$$

Substitute

$$\sin \rho\xi_1 \sin \rho(x - \xi_1) = \tfrac{1}{2}[\cos \rho(x - 2\xi_1) - \cos \rho x],$$

$$U_1 = -\frac{\sigma}{2\rho}\cos \rho x\int_0^x g(\xi_1)d\xi_1 + \frac{\sigma}{2\rho}\int_0^x \cos \rho(x - 2\xi_1)g(\xi_1)d\xi_1.$$

From a lemma by Riemann* it follows that the second term on the right is of the order of $1/\rho^2$.

* Gesammelte Werke, p. 241; Whittaker and Watson, Modern Analysis, 2d ed., p. 166.

Assume

(8)
$$h(x) = \int_0^x g(\xi_1)d\xi_1, \qquad h(1) = h.$$
$$H(x, \sigma, \rho) = \frac{\sigma}{2\rho}\int_0^x \cos \rho(x - 2\xi_1)g(\xi_1)d\xi_1 + \sum_{\nu=2}^{\infty} U_\nu.$$

Then from (6),

(9)
$$y(x) = \sin \rho x - \frac{\sigma h(x)}{2\rho}\cos \rho x + H(x, \sigma, \rho).$$

The function $y(x)$ has already been so determined as to satisfy the first boundary condition $y(0) = 0$, for all values of ρ; it remains to determine the particular values of ρ for which $y(1) = 0$, or the characteristic numbers. This condition becomes:

(10)
$$\tan \rho = \frac{\sigma h}{2\rho} - \frac{H(1, \sigma, \rho)}{\cos \rho} = \phi(\sigma, \rho).$$

This equation can be solved by a method of successive approximations involving a Lipschitz condition of the form
$$| \phi(\sigma, \rho'') - \phi(\sigma, \rho') | < C_\rho | \rho'' - \rho' |.$$

To find an upper bound for C_ρ, compute the partial derivatives $\partial U_\nu / \partial \rho$:
$$\frac{\partial U_\nu}{\partial \rho} = -\frac{\nu}{\rho} U_\nu(x,\sigma, \rho) + \left(\frac{\sigma}{\rho}\right)^\nu \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{\nu-1}} \Bigg[\sum_{\kappa=1}^{\nu} K_\rho(x, \xi_1)$$
$$\cdots K_\rho(\xi_{k-2}, \xi_{k-1}, \rho) \frac{\partial}{\partial \rho} K_\rho(\xi_{k-1}, \xi_k) \cdots K_\rho(\xi_{\nu-1}, \xi_\nu) \Bigg] [\sin \rho\xi_\nu d\xi_1 d\xi_2 \cdots d\xi_\nu]$$
$$+ \left(\frac{\sigma}{\rho}\right)^\nu \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{\nu-1}} K_\rho(x, \xi_1) \cdots K_\rho(\xi_{\nu-1}, \xi_\nu) \xi_\nu \cos \rho\xi_\nu d\xi_1 d\xi_2 \cdots d\xi_\nu.$$

Since
$$\frac{\partial}{\partial \rho} K_\rho(\xi_{\kappa-1}, \xi_\kappa) = (\xi_{\kappa-1} - \xi_\kappa) \cos \rho(\xi_{\kappa-1} - \xi_\kappa) g(\xi_\kappa),$$
$$\left| \frac{\partial}{\partial \rho} K_\rho(\xi_{\kappa-1}, \xi_\kappa) \right| \leq G.$$

Consequently

(11)
$$\left| \frac{\partial}{\partial \rho} U_\nu(x, \sigma, \rho) \right| \leq \frac{1}{(\nu - 1)!} \left(\frac{\sigma G x}{\rho}\right)^{\nu-1} \left[\frac{\sigma G x}{\rho}\left(1 + \frac{1}{\rho} + \frac{1}{\nu}\right) \right].$$

From (8),

(12)
$$\frac{\partial H}{\partial \rho} = \frac{-\sigma}{2\rho^2}\int_0^x \cos \rho(x - 2\xi_1)g(\xi_1)d\xi_1$$
$$- \frac{\sigma}{2\rho}\int_0^x (x - 2\xi_1) \sin \rho(x - 2\xi_1)g(\xi_1)d\xi_1 + \sum_{\nu=2}^{\infty} \frac{\partial U_\nu}{\partial \rho}.$$

It follows from (11), (12), and Riemann's lemma that there exists a constant a_1 such that if ρ is real and greater than 1,

$$\frac{\partial H}{\partial \rho} < \frac{a_1}{\rho^2}, \tag{13}$$

From (10),

$$\frac{\partial \phi}{\partial \rho} = \frac{-\sigma h}{2\rho^2} - \frac{\cos \rho \dfrac{\partial H}{\partial \rho} + H \sin \rho}{\cos^2 \rho}. \tag{14}$$

From (7), (8), a constant a_2 can be found such that if $\rho > 1$,

$$| H(x, \sigma, \rho) | < \frac{a_2}{\rho^2}. \tag{15}$$

If ρ is so chosen that $| \cos \rho | \geqq \frac{1}{2}$, and $| \rho | > 1$, it follows from (13), (14), (15) that for some m,

$$\left| \frac{\partial \phi}{\partial \rho} \right| \leqq \frac{m}{\rho^2}. \tag{16}$$

This inequality holds for $0 \leqq x \leqq 1$, $0 \leqq \sigma \leqq 1$, and ρ real; it leads immediately to the Lipschitz condition desired.

It will be convenient also to calculate $\partial\phi/\partial\sigma$; from (10),

$$\frac{\partial \phi}{\partial \sigma} - \frac{h}{2\rho} = - \frac{\dfrac{\partial H}{\partial \sigma}}{\cos \rho}.$$

If $| \cos \rho | \geqq \frac{1}{2}$,

$$\begin{aligned} \left| \frac{\partial \phi}{\partial \sigma} - \frac{h}{2\rho} \right| &\leqq 2 \left| \frac{\partial H}{\partial \sigma} \right| \\ &\leqq 2 \left| \frac{1}{2\rho} \int_0^x \cos \rho(x - 2\xi_1) g(\xi_1) d\xi_1 + \sum_{\nu=2}^{\infty} \frac{\partial U_\nu}{\partial \sigma} \right|. \end{aligned}$$

Since from (5),

$$\left| \frac{\partial U_\nu}{\partial \sigma} \right| \leqq \frac{\nu}{\rho} \cdot \left(\frac{\sigma}{\rho} \right)^{\nu-1} \frac{(Gx)^\nu}{\nu!} \leqq \frac{Gx}{\rho} \left(\frac{\sigma G x}{\rho} \right)^{\nu-1} \cdot \frac{1}{(\nu-1)!},$$

we have finally,

$$\left| \frac{\partial \phi}{\partial \sigma} - \frac{h}{2\rho} \right| \leqq \frac{a_3}{\rho^2}, \tag{17}$$

where a_3 is a constant; ρ is real and $| \rho | > 1$, $| \cos \rho | \geqq \frac{1}{2}$.

2. **Asymptotic expansion of the characteristic numbers.** In the equation

$$\tan \rho = \phi(\sigma, \rho)$$

put $\rho = \kappa\pi + \epsilon_\kappa$; the equation becomes

$$\tan \epsilon_\kappa = \phi(\sigma, \kappa\pi + \epsilon_\kappa). \tag{18}$$

Let δ_κ be the interval $\kappa\pi - (\pi/3) \leqq \rho \leqq \kappa\pi + (\pi/3)$. From the inequalities already established it is seen immediately that N can be chosen so large that if $\kappa > N$, and if ρ_1, ρ_2 are in the same interval δ_κ, the following inequalities are satisfied:

$$(19) \qquad \begin{aligned} &(a) \ | \phi(\sigma, \rho_1) | < \frac{C}{(\kappa - 1)\pi}, \\ &(b) \ | \phi(\sigma, \rho_2) - \phi(\sigma, \rho_1) | < \frac{m | \rho_2 - \rho_1 |}{[(\kappa - 1)\pi]^2}, \\ &(c) \ \frac{4m}{[(\kappa - 1)\pi]^2} < 1, \\ &(d) \ \frac{2C}{(\kappa - 1)\pi} \cdot \frac{1}{1 - \dfrac{4m}{[(\kappa - 1)\pi]^2}} < \frac{\pi}{6}. \end{aligned}$$

Condition (b) follows from (16) by integration. Let ϵ', ϵ'', $\cdots$, $\epsilon^{(\nu)}$ be defined by the equations

$$(20) \qquad \begin{aligned} \tan \epsilon' &= \phi(\sigma, \kappa\pi), \\ \tan \epsilon'' &= \phi(\sigma, \kappa\pi + \epsilon'), \\ &\cdots\cdots\cdots \\ \tan \epsilon^{(\nu)} &= \phi(\sigma, \kappa\pi + \epsilon^{(\nu-1)}). \end{aligned}$$

In solving this system, observe that if $| \mu | < 1$,

$$(21) \qquad \arctan \mu = \int_0^\mu \frac{dx}{1 + x^2}, \qquad | \arctan \mu | < \frac{| \mu |}{1 - \mu^2}.$$

From this inequality we obtain:

$$| \epsilon' | < \frac{| \phi(\sigma, \kappa\pi) |}{1 - \phi(\sigma, \kappa\pi)^2}.$$

From (19); (a) and (d),

$$| \phi(\sigma, \kappa\pi) | < \frac{\pi}{12}.$$

Consequently

$$| \phi^2(\sigma, \kappa\pi) | < \tfrac{1}{2},$$

$$(22) \qquad | \epsilon' | < 2 | \phi(\sigma, \kappa\pi) | < \frac{\pi}{6}.$$

It follows that $(\kappa\pi + \epsilon')$ lies in the interval δ_κ, and ϵ'' can be determined from (20).

Applying the method of induction, we shall assume that $(\kappa\pi + \epsilon')$, $(\kappa\pi + \epsilon'') \cdots (\kappa\pi + \epsilon^{(\nu)})$ lie in the same interval δ_κ; in addition, assume that for $\mu < \nu$,

$$(23) \qquad |\epsilon^{(\mu+1)} - \epsilon^{(\mu)}| < \frac{2C}{(\kappa-1)\pi}\left[\frac{4m}{[(\kappa-1)\pi]^2}\right]^{\mu},$$

$$\tan[\epsilon^{(\nu+1)} - \epsilon^{(\nu)}] = \frac{\phi(\sigma, \kappa\pi + \epsilon^{(\nu)}) - \phi(\sigma, \kappa\pi + \epsilon^{(\nu-1)})}{1 + \phi(\sigma, \kappa\pi + \epsilon^{(\nu)})\phi(\sigma, \kappa\pi + \epsilon^{(\nu-1)})}.$$

From (19) (b), (d), and from (23),

$$|\tan[\epsilon^{(\nu+1)} - \epsilon^{(\nu)}]| < \frac{2m}{[(\kappa-1)\pi]^2}\cdot\frac{2C}{(\kappa-1)\pi}\left[\frac{4m}{[(\kappa-1)\pi]^2}\right]^{\nu-1}.$$

From (19) (c), (d), and from (21),

$$[\epsilon^{(\nu+1)} - \epsilon^{(\nu)}] < \frac{2C}{(\kappa-1)\pi}\left[\frac{4m}{[(\kappa-1)\pi]^2}\right]^{\nu},$$

$$|\epsilon^{(\nu+1)}| = |\epsilon' + (\epsilon'' - \epsilon') + \cdots + (\epsilon^{(\nu+1)} - \epsilon^{(\nu)})|$$

$$(24) \qquad < \frac{2C}{(\kappa-1)\pi}\left[1 + \frac{4m}{[(\kappa-1)\pi]^2} + \cdots + \left[\frac{4m}{[(\kappa-1)\pi]^2}\right]^{\nu}\right]$$

$$< \frac{\pi}{6},$$

from (19), (d). Consequently the inequalities (23), (24) hold for $\nu = 1$, and ϵ''' can be determined from (20), while $(\kappa\pi + \epsilon')$, $(\kappa\pi + \epsilon'')$ lie in δ_κ; $(\kappa\pi + \epsilon''')$ lies in δ_κ, hence $\epsilon^{(4)}$ can be determined, and $(\kappa\pi + \epsilon^{(4)})$ lies in δ_κ, etc. By the principle of induction (23) and (24) hold for $\nu = 1, 2, 3, \cdots, \nu, \cdots$; from (23) the series

$$\epsilon_\kappa = \epsilon' + (\epsilon'' - \epsilon') + (\epsilon''' - \epsilon'') + \cdots$$

converges absolutely and uniformly with respect to σ, for $0 \leqq \sigma \leqq 1$; consequently ϵ_κ is a continuous function of σ.

To verify that ϵ_κ satisfies (18), observe that in the identity

$$\begin{aligned}\tan\epsilon_\kappa - \phi(\sigma, \kappa\pi + \epsilon_\kappa) = {} & [\tan\epsilon_\kappa - \tan\epsilon^{(\nu+1)}] \\ & + [\tan\epsilon^{(\nu+1)} - \phi(\sigma, \kappa\pi + \epsilon^{(\nu)})] \\ & + [\phi(\sigma, \kappa\pi + \epsilon^{(\nu)}) - \phi(\sigma, \kappa\pi + \epsilon_\kappa)],\end{aligned}$$

the second term on the right vanishes, while the first and last terms can be shown to approach zero, as ν becomes infinite, with the aid of the inequalities (19) and (23). Since the difference on the left is independent of ν, this difference must vanish.

The asymptotic expansion of ϵ_κ can be determined with the aid of the inequalities established above.

$$\begin{aligned}
|\epsilon_\kappa - \epsilon'| &= |(\epsilon'' - \epsilon') + (\epsilon''' - \epsilon'') + \cdots| \\
&< \frac{2C}{(\kappa - 1)\pi}\left[\frac{4m}{[(\kappa - 1)\pi]^2} + \cdots\right] \\
&< \frac{\pi}{6}\cdot\frac{4m}{[(\kappa - 1)\pi]^2},
\end{aligned}$$

$$\begin{aligned}
\epsilon' &= \arctan \phi(\sigma, \kappa\pi) \\
&= \phi(\sigma, \kappa\pi) + C_3[\phi(\sigma, \kappa\pi)]^3 \pm \cdots \\
&= \frac{h\sigma}{2\kappa\pi} - \frac{H}{\cos \kappa\pi} + \cdots
\end{aligned}$$

$$\begin{aligned}
\left|\epsilon_\kappa - \frac{h\sigma}{2\kappa\pi}\right| &= \left|(\epsilon_\kappa - \epsilon') + \left(\epsilon' - \frac{h\sigma}{2\kappa\pi}\right)\right| \\
&< \frac{\alpha}{\kappa^2\pi^2},
\end{aligned}$$

$$\epsilon_\kappa = \frac{h\sigma}{2\kappa\pi} + \frac{\overline{H}(\sigma, \kappa)}{\kappa^2\pi^2}, \tag{25}$$

where $\overline{H}$ is a function less than some constant H for all values of $\kappa > N$, and for $0 \leqq \sigma \leqq 1$. For large values of κ the derivative $d\rho_\kappa/d\sigma$ exists. For suppose σ_1, σ_2 two values of σ in the interval $0 \leqq \sigma \leqq 1$, ρ_1, ρ_2 the corresponding values of ρ for a given κ.

From the equations

$$\tan \rho_1 = \phi(\sigma_1, \rho_1), \qquad \tan \rho_2 = \phi(\sigma_2, \rho_2),$$

$$\frac{\tan \rho_2 - \tan \rho_1}{\rho_2 - \rho_1} = [1 + \tan \rho_1 \tan \rho_2]\frac{\tan(\rho_2 - \rho_1)}{\rho_2 - \rho_1},$$

we obtain

$$\frac{\rho_2 - \rho_1}{\sigma_2 - \sigma_1} = \frac{\dfrac{\phi(\sigma_2, \rho_1) - \phi(\sigma_1, \rho_1)}{\sigma_2 - \sigma_1}}{[1 + \phi(\sigma_1, \rho_1)\phi(\sigma_2, \rho_2)]\dfrac{\tan(\rho_2 - \rho_1)}{\rho_2 - \rho_1} - \dfrac{\phi(\sigma_2, \rho_2) - \phi(\sigma_2, \rho_1)}{\rho_2 - \rho_1}}.$$

Let the difference $\sigma_2 - \sigma_1$ approach zero; the partial derivatives approached on the right exist and are continuous in ρ, σ; consequently

$$\begin{aligned}
\frac{d\rho}{d\sigma} &= \frac{\dfrac{\partial\phi}{\partial\sigma}}{1 + \phi^2 - \dfrac{\partial\phi}{\partial\rho}}, \\
&= \frac{\partial\phi}{\partial\sigma}\left[1 + \frac{\dfrac{\partial\phi}{\partial\rho} - \phi^2}{1 + \phi^2 - \dfrac{\partial\phi}{\partial\rho}}\right].
\end{aligned}$$

Consequently from (11),

$$\frac{d\rho}{d\sigma} = \frac{h}{2\rho} + \frac{K_\rho(\sigma)}{\rho^2}, \tag{26}$$

where K_ρ is a function of ρ, σ, bounded for large values of ρ.

3. **The characteristic functions.** In equation (9),

$$y(x) = \sin \rho x - \frac{\sigma h(x)}{2\rho} \cos \rho x + H(x, \sigma, \rho), \tag{9}$$

substitute $\rho = \kappa\pi + \epsilon_\kappa$:

$$\sin \rho x = \sin \kappa\pi x + \epsilon_\kappa x \cos \kappa\pi x + \epsilon_\kappa{}^2 \psi_1(x, \kappa, \sigma),$$

$$\cos \rho x = \cos \kappa\pi x + \epsilon_\kappa \psi_2(x, \kappa, \sigma),$$

where ψ_1, ψ_2 and hereafter ψ_ν will denote functions of x, κ, σ continuous in x and less than some constant C_ν for $\kappa > N$, and $0 \leqq \sigma \leqq 1$.

Equation (9) becomes,

$$y(x) = \sin \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \cos \kappa\pi x + \frac{\psi_3}{\kappa^2}. \tag{9'}$$

Equations (9) and (9′) are identical for $0 \leqq \sigma \leqq 1$. In (9) compute $\partial y/\partial\sigma$:

$$\begin{aligned} \frac{\partial y}{\partial \sigma} &= \frac{\partial H}{\partial \sigma} - \frac{h(x)}{2\rho} \cos \rho x \\ &\quad + \frac{d\rho}{d\sigma}\left[x \cos \rho x + \frac{\sigma h(x)}{2\rho^2} \cos \rho x + \frac{x\sigma h(x)}{2\rho} \sin \rho x + \frac{\partial H}{\partial \rho} \right] \\ &= \frac{xh - h(x)}{2\kappa\pi} \cos \kappa\pi x + \frac{\psi_4}{\kappa^2}. \end{aligned} \tag{27}$$

From (9′),

$$\frac{\partial y}{\partial \sigma} = \frac{xh - h(x)}{2\kappa\pi} \cos \kappa\pi x + \frac{1}{\kappa^2} \frac{\partial \psi_3}{\partial \sigma}.$$

Comparing results,

$$\frac{\partial \psi_3}{\partial \sigma} = \psi_4.$$

From (9′) compute

$$\int_0^1 y^2(x) dx$$

$$\begin{aligned} \int_0^1 y^2(x) dx &= \int_0^1 \left[\sin^2 \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \sin 2\kappa\pi x + \frac{\psi_5}{\kappa^2} \right] dx \\ &= \frac{1}{2} + \frac{\psi_6}{\kappa^2}, \end{aligned} \tag{28}$$

since from Riemann's Lemma,

$$\left|\int_0^1 \frac{\sigma}{2\kappa\pi}[xh - h(x)] \sin 2\kappa\pi x dx\right| < \frac{c}{\kappa^2}.$$

Also,

$$\frac{d}{d\sigma}\int_0^1 y^2(x)dx = 2\int_0^1 y(x)\left[\frac{1}{2\kappa\pi}(xh - h(x)) \cos \kappa\pi x + \frac{\psi_4}{\kappa^2}\right]dx,$$

applying Riemann's Lemma again, we obtain the result

$$\left|\frac{d}{d\sigma}\int_0^1 y^2(x)dx\right| < \frac{c'}{\kappa^2},$$

where c' as well as c is a constant independent of κ if $\kappa > N$. From (28)

$$\left|\frac{\partial\psi_6}{\partial\sigma}\right| < c'; \qquad \frac{\partial\psi_6}{\partial\sigma} = \psi_7. \tag{29}$$

Each of the functions $y_\kappa(x)$ is *normalized* by division by the function, of σ,

$$\sqrt{\int_0^1 y_\kappa^2(x)dx}.$$

$$\frac{1}{\int_0^1 y_\kappa^2(x)dx} = \frac{1}{\frac{1}{2} + \frac{\psi_6}{\kappa^2}} = 2 - \frac{\psi_6}{\kappa^2\left(\frac{1}{2} + \frac{\psi_6}{\kappa^2}\right)} = 2 + \frac{\psi_8}{\kappa^2},$$

$$\sqrt{2 + \frac{\psi_8}{\kappa^2}} = \sqrt{2} + \sqrt{2}\left[\sqrt{1 + \frac{\psi_8}{2\kappa^2}} - 1\right].$$

If $|z| < 1$,

$$\sqrt{1 + z} - 1 = \frac{z}{2} - \frac{1\cdot 1}{2\cdot 4}z^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}z^3 \pm \cdots,$$

$$|\sqrt{1 + z} - 1| < \frac{|z|}{2}[1 + |z| + |z|^2 + \cdots]$$

$$< \frac{1}{2}\frac{|z|}{1 - |z|}.$$

If the constant N has been chosen sufficiently large,

$$\left|\frac{\psi_6}{2\kappa^2}\right| < \frac{1}{2},$$

and accordingly

$$\sqrt{2 + \frac{\psi_8}{\kappa^2}} = \sqrt{2} + \frac{\psi_9}{\kappa^2}.$$

$$
(30)\quad \begin{aligned}\bar{y}(x) &= \frac{y(x)}{\sqrt{\int_0^1 y^2(x)dx}} = y(x)\left[\sqrt{2} + \frac{\psi_9}{\kappa^2}\right] \\ &= \sqrt{2}\left\{\sin \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \cos \kappa\pi x + \frac{\psi_{10}}{\kappa^2}\right\}.\end{aligned}
$$

$$
(31)\quad \begin{aligned}\frac{\partial \bar{y}(x)}{\partial \sigma} &= \frac{\sqrt{\int_0^1 y^2(x)dx}\,\dfrac{\partial y(x)}{\partial \sigma} - \dfrac{y(x)}{\sqrt{\int_0^1 y^2(x)dx}}\displaystyle\int_0^1 y(x)\frac{\partial y(x)}{\partial \sigma}dx}{\displaystyle\int_0^1 y^2(x)dx} \\ &= \sqrt{2}\left\{\frac{xh - h(x)}{2\kappa\pi}\cos \kappa\pi x + \frac{\psi_{11}}{\kappa^2}\right\}.\end{aligned}
$$

Comparing (30), (31), we obtain the result:

$$
(32)\qquad \frac{\partial \psi_{10}}{\partial \sigma} = \psi_{11}.
$$

Hence, finally, the normalized functions $y_\kappa(x)$ satisfying the system (2) are given asymptotically in the form

$$
y_\kappa(x) = \sqrt{2}\left\{\sin \kappa\pi x + \frac{\sigma}{2\kappa\pi}[xh - h(x)] \cos \kappa\pi x + \frac{\psi(x, \sigma, \kappa)}{\kappa^2}\right\},
$$

$$
(32)\qquad \left|\frac{\partial \psi(x, \sigma, \kappa)}{\partial \sigma}\right| < K,
$$

where K is independent of κ, $\sigma < 1$ for $\kappa > N$.

4. **Nature of the solutions for small values of ρ.** In the preceding discussion the parameter ρ was assumed larger than a constant N'; to complete the discussion it is necessary to study the functions $y(x)$ for small values of ρ. For this purpose write the equation of (2) in the form

$$
(2')\qquad \frac{d^2y}{dx^2} = [\sigma g(x) - \lambda]y.
$$

The solutions of the homogeneous equation obtained by equating the left-hand member to zero are 1 and x; the corresponding integral equation becomes,

$$
y(x) = \alpha + \beta x + \int_0^x (x - \xi)[\sigma g(\xi) - \lambda]y(\xi)d\xi,
$$

since $y(0) = 0$, $\alpha = 0$; for convenience assume $\beta = 1$,

$$
y(x) = x + \int_0^x (x - \xi)[\sigma g(\xi) - \lambda]y(\xi)d\xi.
$$

For values of λ less than or equal to N'^2 the kernel is bounded, and continuous and differentiable with respect to σ, λ; the series expansion of the solution of this integral equation converges uniformly as before for $0 \leq \sigma \leq 1$, $|\lambda| \leq N'^2$. Consequently the solution $y(x)$ is continuous in λ, σ, and as before the partial derivatives with respect to these parameters can be computed from the series expansion.

If $\partial y/\partial\lambda$ has continuous first and second partial derivatives with respect to x, such that

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda}\left(\frac{\partial y}{\partial x}\right),$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda}\left(\frac{\partial^2 y}{\partial x^2}\right),$$

then from the equation

$$(2') \qquad y'' + (\lambda - \sigma g(x))y = 0,$$

we obtain:

$$(33) \qquad \frac{d^2}{dx^2}\left(\frac{\partial y}{\partial \lambda}\right) + (\lambda - \sigma g(x))\frac{\partial y}{\partial \lambda} = -y.$$

Let $y(x)$ be a solution of (2′) vanishing for $x = 0$, and suppose $\bar{y}(x)$ any solution not vanishing at the origin for any value of σ. The solution of (33) becomes,

$$\frac{\partial y}{\partial \lambda} = c_1 y(x) + c_2 \bar{y}(x) + \frac{1}{y\bar{y}' - \bar{y}y'}\int_0^x [\bar{y}(x)y(\xi) - y(x)\bar{y}(\xi)][-y(\xi)]d\xi.$$

When $x = 0$, $\partial y/\partial\lambda = 0$, $y(0) = 0$, hence $c_2 = 0$.

$$\frac{\partial y}{\partial \lambda} = c_1 y(x) - \frac{1}{y\bar{y}' - \bar{y}y'}\int_0^x [\bar{y}(x)y(\xi) - y(x)\bar{y}(\xi)]y(\xi)d\xi.$$

The conditions $\partial^2 y/\partial x\partial\lambda = \partial^2 y/\partial\lambda\partial x$, etc., are easily seen to be satisfied. Suppose λ a characteristic number; then $y(1) = 0$. Put $x = 1$.

$$\left[\frac{\partial y}{\partial \lambda}\right]_{x=1} = \frac{-\bar{y}(1)}{y\bar{y}' - \bar{y}y'}\int_0^1 y^2(\xi)d\xi.$$

Since $y(1) = 0$, $\bar{y}(1) \neq 0$, hence

$$\left[\frac{\partial y}{\partial \lambda}\right]_{x=1} \neq 0.$$

It follows immediately that $d\lambda/d\sigma$ can be computed from the equation

$$\left[\frac{\partial y}{\partial \sigma}\right]_{x=1} + \left[\frac{\partial y}{\partial \lambda}\frac{d\lambda}{d\sigma}\right]_{x=1} = 0$$

and is finite for $\rho < N'$.

Consequently if σ_1, σ_2 lie in the interval $0 \leq \sigma \leq 1$, $\sigma_1 < \sigma_2$,

$$\begin{aligned} y_\kappa(x, \sigma_2) - y_\kappa(x, \sigma_1) &\leq \int_{\sigma_1}^{\sigma_2} \left| \frac{\partial y_\kappa}{\partial \sigma} + \frac{\partial y_\kappa}{\partial \lambda} \frac{d\lambda}{d\sigma} \right| d\sigma \\ &< A_\kappa \,|\, \sigma_2 - \sigma_1 \,|, \end{aligned}$$

where A_κ is independent of σ, for $0 \leq \sigma \leq 1$.

Since N' is finite, some A_κ is equal to or greater than any of the others; call this one A:

$$|\, y_\kappa(x, \sigma_2) - y_\kappa(x, \sigma_1) \,| < A \,|\, \sigma_2 - \sigma_1 \,| \tag{34}$$

$$\rho_\kappa < N', \qquad 0 \leqq \sigma_1 < \sigma_2 \leqq 1.$$

It has been seen that $y_\kappa(x, \sigma)$ is continuous in σ; it follows that $y_\kappa(x, \sigma)$ vanishes just $(\kappa - 1)$ times in the interval $0 < x < 1$. For this is true when $\sigma = 0$; as σ increases from 0 to 1 the end-points of the curve $y = y_\kappa(x)$ remain fixed, and consequently zeros can enter or disappear only if for at least one value of σ the curve becomes tangent to the x-axis. At the point of tangency $y = y' = 0$; since $y(x)$ satisfies the differential equation (2) $y(x)$ vanishes identically. But from (32) this is seen to be impossible for very large values of κ; suppose N so large that for $\kappa \geq N$ $y_\kappa(x, \sigma)$ does not vanish identically for any value of σ between 0 and 1. Then $y_\kappa(x, \sigma)$ has the same number of zeros for $\sigma = 1$ as for $\sigma = 0$, or $(\kappa - 1)$ zeros. Now for $\kappa \leq N$ the characteristic functions can be so ordered that $y_\kappa(x, 1)$ has one more zero than $y_{\kappa-1}(x, 1)$; assuming the functions y_κ in this order, y_{N-1} has $(N - 2)$ zeros, y_{N-2} has $(N - 3)$ zeros,—hence for $\kappa \leq N$ as well as for $\kappa > N$, y_κ has just $(\kappa - 1)$ zeros in the interval

$$0 < x < 1.$$

The equation and inequality (32), together with (34), are sufficient for an immediate application of the theorem of Professor Birkhoff, with the aid of which the closure of the set of normalized functions satisfying (1) is established.